

Performance Measures for Parameter Extraction of Sensor Array Point Targets using the Discrete Chirp Fourier Transform

Nayda Santiago^a, Cesar A Aceros Moreno^a and Domingo Rodriguez^a

^a Automated Information Processing Laboratory - Electrical and Computer Engineering Department, University of Puerto Rico at Mayagüez
Mayagüez, Puerto Rico 00681-9042 , Mayagüez, PR

ABSTRACT

This work presents a new methodology for the formulation of discrete chirp Fourier transform (DCFT) algorithms and it discusses performance measures pertaining to the mapping of these algorithms to hardware computational structures (HCS) as well as the extraction of chirp rate estimation parameters of multicomponent nonstationary signals arriving from point targets. The methodology centers on the use of Kronecker products algebra, a branch of finite dimensional multilinear algebra, as a language to present a canonical formulation of the DCFT algorithm and its associated properties. The methodology also explains how to search for variants of this canonical formulation that contribute to enhance the mapping process to a target HCS. The parameter extraction technique uses time-frequency properties of the DCFT in a modeled delay-Doppler synthetic aperture radar (SAR) remote sensing and surveillance environment to treat multicomponent return signals of prime length, with additive Gaussian noise as background clutter, and extract associated chirp rate parameters. The fusion of time-frequency information, acquired from transformed chirp or linear frequency modulated (FM) signals using the DCFT, with information obtained when the signals are treated using the discrete ambiguity function acting as point target response, point spread function, or impulse response, is used to further enhance the estimation process.

For the case of very long signals, parallel algorithm implementations have been obtained on cluster computers. A theoretical computer performance analysis was conducted on the cluster implementation based on a methodology that applies well-defined design of experiments methods to the identification of relations among different levels in the process of mapping computational operations to high-performance computing systems. The use of statistics for identification of relationships among factors has formalized the search for solutions to the mapping problem and this approach allows unbiased conclusions about results.

Keywords: Computer performance evaluation, high performance computing, performance metrics, discrete chirp Fourier transform, discrete ambiguity function

1. INTRODUCTION

This article presents some results of research works conducted on computational signal algebra techniques for the design and development of discrete chirp Fourier transform algorithms and the use of these algorithms in the estimation of point target phase parameters of multicomponent chirp signals in active remote sensing applications such as synthetic aperture radar (SAR) signal-based information processing. For the algorithm design stage of the work, the discrete chirp Fourier transform (DCFT) was first analyzed from the point of view of the discrete Fourier transform of a modulated signal. The general steps involved in the formulation of a Fast Transform are shown on Figure 1. The formulation of the Transform as an operator is the first step, from this we have to express the transform as a matrix representation, then the matrix representation is decomposed in several product of sparse matrices. And finally the implementation in computational structures allow to see how the proposed formulation works.

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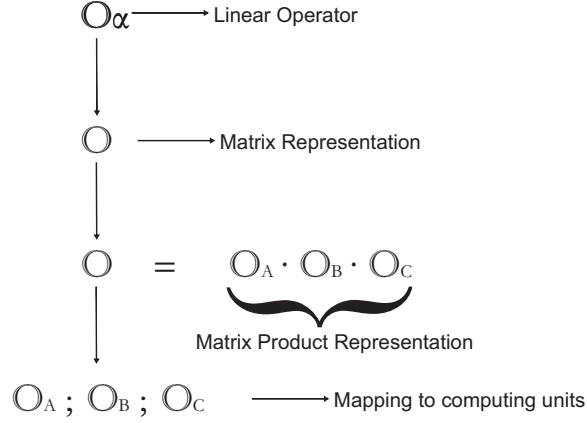


Figure 1. Stages of an Algorithm Development

1.1. Continuous Time Chirp Signal

Chirp signals are prevalent in many engineering and scientific applications such as radar, sonar, and communications. A continuous real chirp signal, with amplitude A , is defined as follows:

$$x(t) = A \cos \left[\frac{1}{2}at^2 + bt \right], \quad t \in \mathbb{R}, \quad \begin{cases} b \triangleq \text{Initial frequency or carrier frequency} \\ a \triangleq \text{Chirp rate} \end{cases} \quad (1)$$

From Equation 1, an instantaneous angular frequency is obtained by taking the time derivative of the phase of the sinusoidal signal resulting in $\frac{d\theta(t)}{dt} = \dot{\theta}(t) = at + b$. An analytic representation of the continuous chirp signal may be formulated as follows:

$$x_a(t) = A \exp(j\frac{1}{2}at^2 + bt) = A \exp(j\frac{1}{2}at^2) \exp(jbt), \quad (2)$$

where $\exp(j\frac{1}{2}at^2) = d(t)$ is termed the frequency rate signal factor. The Fourier transform of the signal $x_a(t)$ is given by the expression $X_a(\Omega) = AD(\Omega) \star \delta(\Omega - b)$. Here, the symbol " \star " denotes linear convolution, and $D(\Omega)$ is the Fourier transform of the frequency rate signal factor.

1.2. Discrete Time Chirp Signal

The discrete version of a real chirp signal, with amplitude A and sampled at a rate of F_s samples per second, is given by the following expression:

$$x[n] = A \cos \left[\frac{1}{2}a(n \cdot T_s)^2 + b(n \cdot T_s) \right], \quad n \in \mathbb{Z}, \quad \begin{cases} n \triangleq \text{Number of sample} \\ F_s = \frac{1}{T_s} \triangleq \text{Sampling frequency} \\ b \triangleq \text{Initial frequency or carrier frequency} \\ a \triangleq \text{Chirp rate} \end{cases} \quad (3)$$

Following the analytic representation of a continuous chirp signal, and after describing a discrete version of a real chirp signal, a sampled version of a finite complex chirp signal can be defined in the same fashion. A finite discrete complex chirp signal of length N is defined by the following expression:

$$x[n] = A_0 W_N^{-(k_0 n + l_0 n^2)}, \quad n \in \mathbb{Z}_N, \quad \begin{cases} n \triangleq \text{Number of sample} \\ k_0 : \text{Carrier frequency} \\ l_0 : \text{Chirp rate} \\ A_0 : \text{Amplitude} \end{cases} \quad (4)$$

Here, $W_N^a = e^{-j\frac{2\pi}{N}a}$ and j is $\sqrt{-1}$. The real part for a complex chirp signal is depicted in Figure 2.

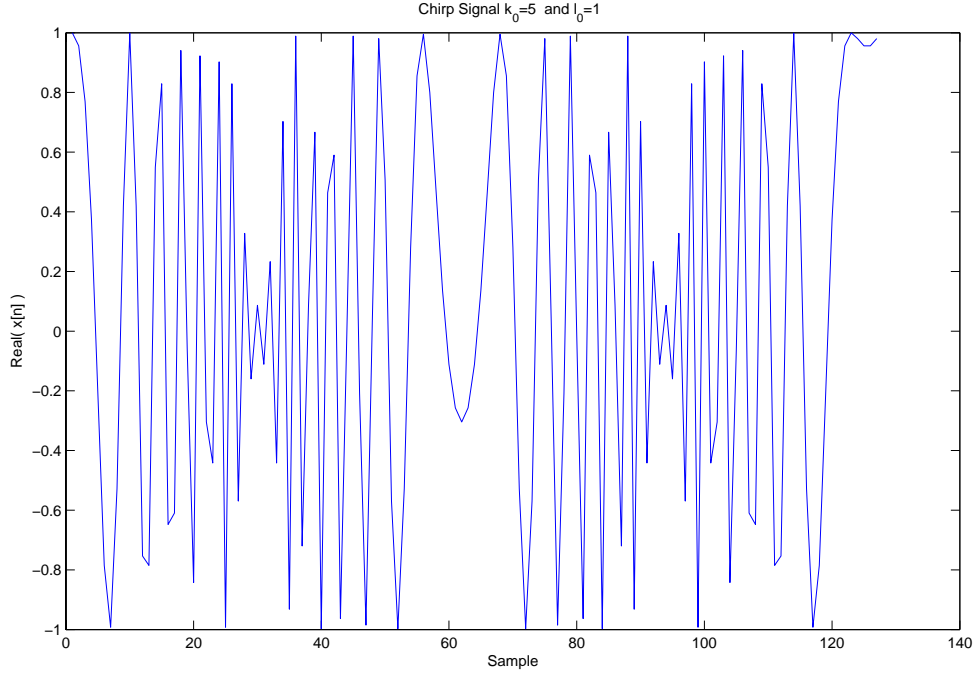


Figure 2. Chirp Signal

1.3. Multicomponent Discrete Time Chirp Signal

A chirp signal resulting from several individual return chirp signals is termed a multicomponent chirp return signal. The mathematical expression for K components of a multicomponent chirp signal is defined as

$$x[n] = \sum_{i=0}^{K-1} A_i W_N^{-(k_i n + l_i n^2)} + \omega[n], \quad n \in \mathbb{Z}_N, \quad \left\{ \begin{array}{l} n \triangleq \text{Number of sample} \\ k_i : i^{\text{th}} \text{ Carrier Frequency} \\ l_i : i^{\text{th}} \text{ Chirp Rate} \\ A_i : i^{\text{th}} \text{ Amplitude} \end{array} \right. \quad (5)$$

2. THE DISCRETE CHIRP FOURIER TRANSFORM

Nowadays, the methods and tools for time-frequency signal analysis are becoming more prevalent due to advances in computational methods and signal processing techniques. A signal is considered a time-frequency signal or a non-stationary signal, if its spectral characteristics change with time. A promising mathematical tool is the 1-D Discrete Chirp Fourier Transform (DCFT). Chirp signals are widely used in radar, sonar, and communications applications.¹ The computational complexity DCFT is rather involved and it gets aggravated when the DCFT is extended to multidimensional domains. The main objective of this work is to present mathematical formulation of algorithms for time-frequency signal analysis of the 1-D DCFT. The work uses theoretic properties of the input and output indexing sets, which are treated as abelian groups, in order to identify a proper mathematical framework for the formulation of the algorithms. The numeric computation and scientific visualization MATLAB® software package has been used for the development of the algorithms. The 1-D DCFT¹ is defined as follows:

$$X_c[k, l] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] W_N^{ln^2 + kn}, \quad 0 \leq k, l \leq N-1 \quad (6)$$

Two parameters, l and k , are identified in Equation 6 which match the chirp rate l_0 and constant frequency value k_0 of a single quadratic chirp at the mainlobe. The k corresponds to the standard spectral parameter of the discrete Fourier transform (DFT). The l parameter corresponds to the frequency rate parameter. From the Equation 6, it is possible to express the following representation as an indexing scheme:

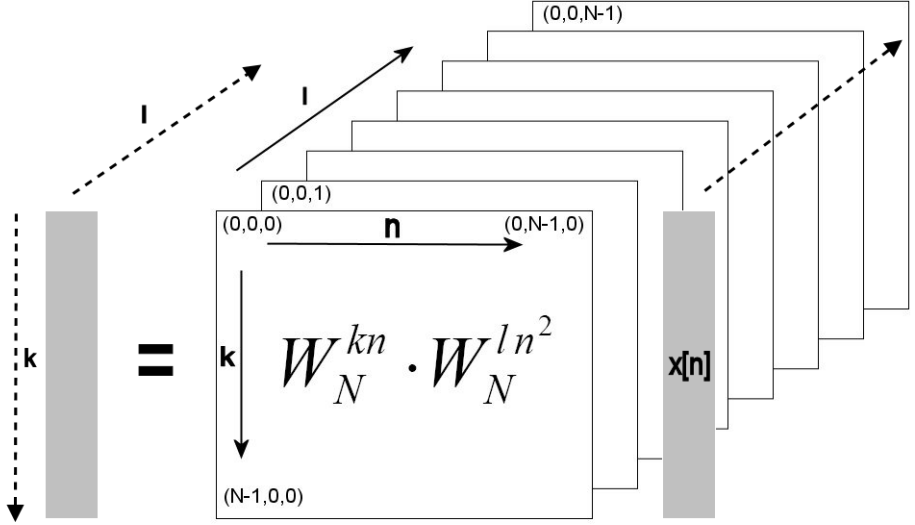


Figure 3. Indexation Scheme

As an example of direct computation, Equation 6 is expanded for case of $N = 8$:

$$\begin{aligned}
 Xc[k, l] = & x[0] + W_N^l W_N^k x[1] + W_N^{4l} W_N^{2k} x[2] \\
 & + W_N^{9l} W_N^{3k} x[3] + W_N^{16l} W_N^{4k} x[4] \\
 & + W_N^{25l} W_N^{5k} x[5] + W_N^{36l} W_N^{6k} x[6] \\
 & + W_N^{49l} W_N^{7k} x[7]
 \end{aligned} \tag{7}$$

3. THE DCFT AS A TOOL FOR PHASE PARAMETERS ESTIMATION

Modifying the parameters of the signal $x[n]$ which appear in Equation 5, results in the signals $x_4[n]$, $x_5[n]$, $x_6[n]$, $x_7[n]$, $x_8[n]$, and $x_9[n]$, which, after applying the DCFT transform, as defined by Equation 6, results in the transformed signals $X_{C_4}[k, l]$, $X_{C_5}[k, l]$, $X_{C_6}[k, l]$, $X_{C_7}[k, l]$, $X_{C_8}[k, l]$, and $X_{C_9}[k, l]$, respectively. The spectral results are depicted in Figures 4, 5, 6, 7, 8, and 9, respectively. These Figures show the magnitude of the DCFT in the output space ($[k, l]$ plane). The estimated chirp parameters are clearly identified. Each (k_n, l_n) component is represented for the multicomponent chirp signal modeling radar returns.

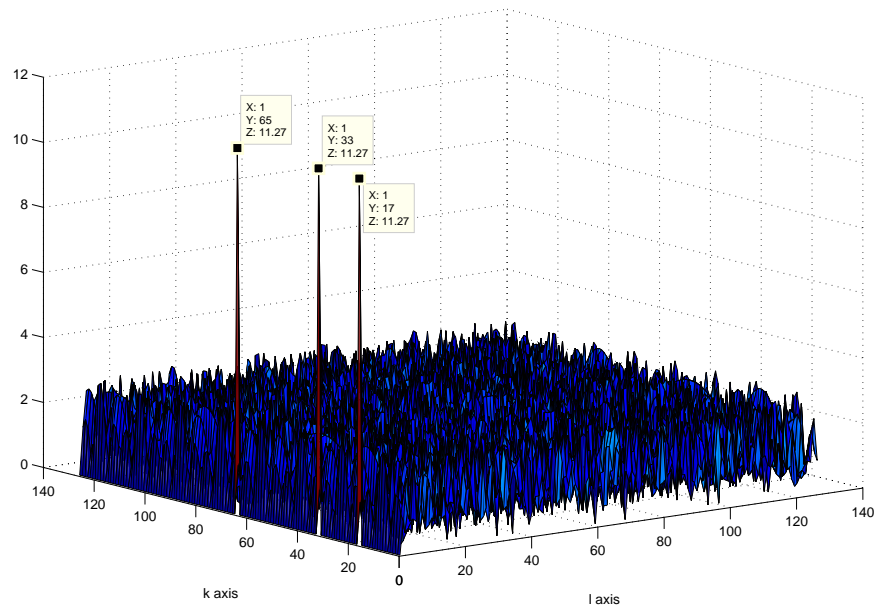


Figure 4. $X_{C_4} - (k_0 = 16, l_0 = 0); (k_1 = 32, l_1 = 0); (k_2 = 64, l_2 = 0)$

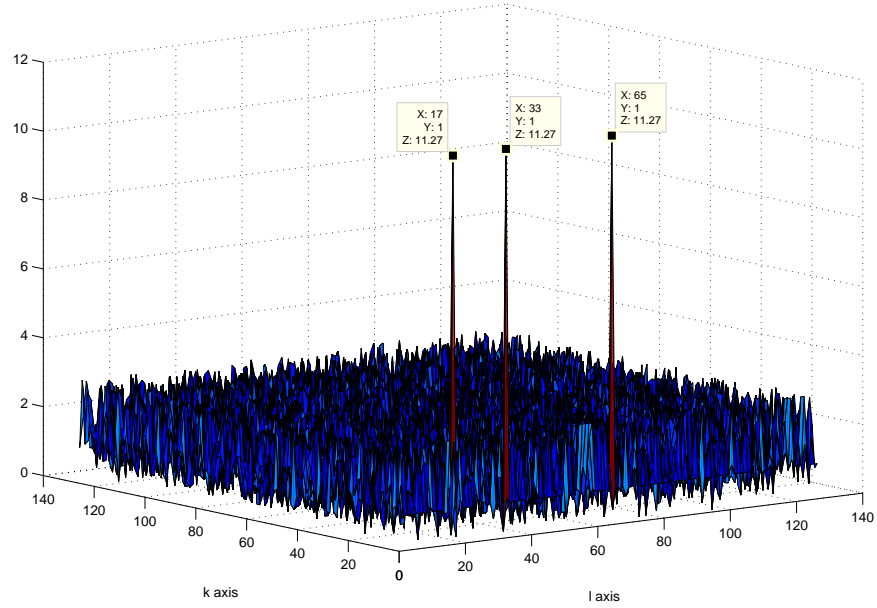


Figure 5. $X_{C_5} - (k_0 = 0, l_0 = 16); (k_1 = 0, l_1 = 32); (k_2 = 0, l_2 = 64)$

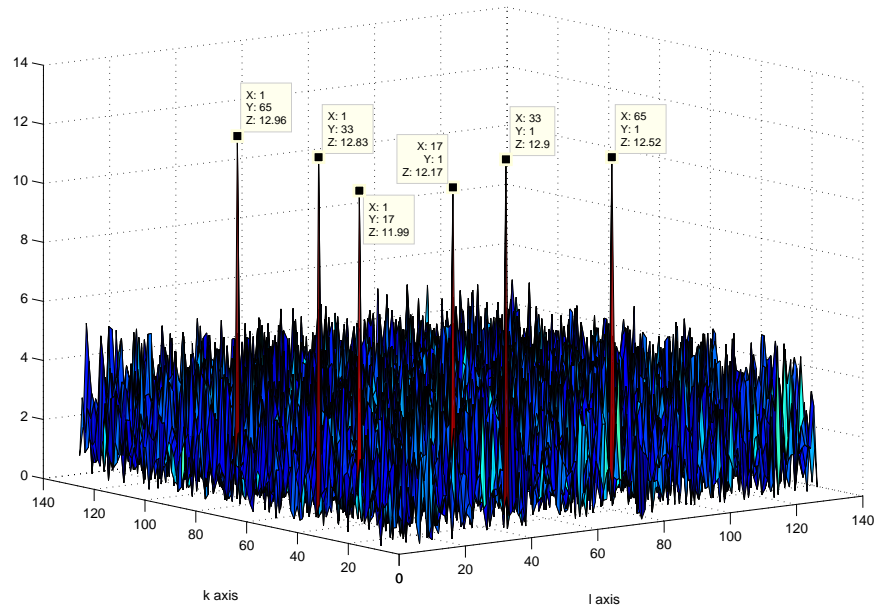


Figure 6. $X_{C_6} - (k_0 = 16, l_0 = 0); (k_1 = 32, l_1 = 0); (k_2 = 64, l_2 = 0); (k_0 = 0, l_0 = 16); (k_1 = 0, l_1 = 32); (k_2 = 0, l_2 = 64)$

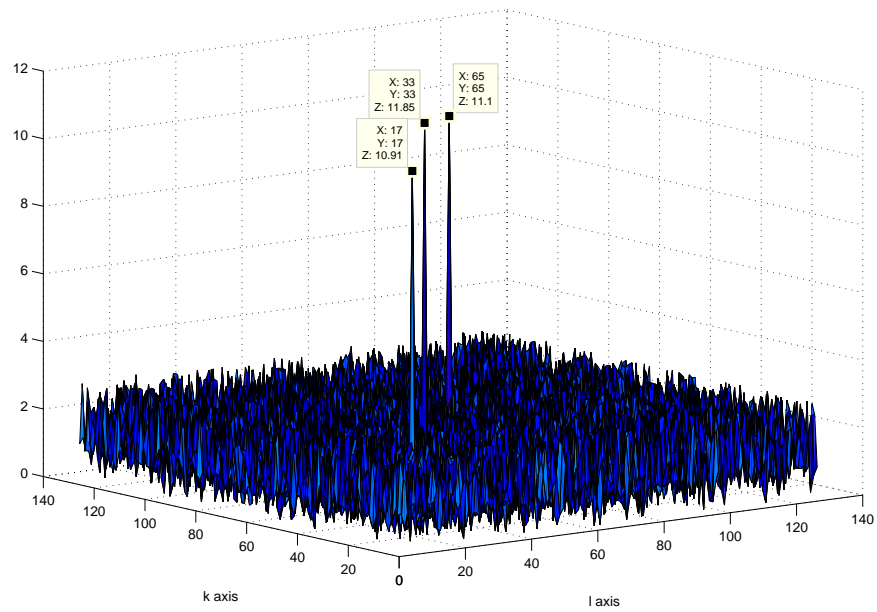


Figure 7. $X_{C_7} - (k_0 = 16, l_0 = 16); (k_1 = 32, l_1 = 32); (k_2 = 64, l_2 = 64)$

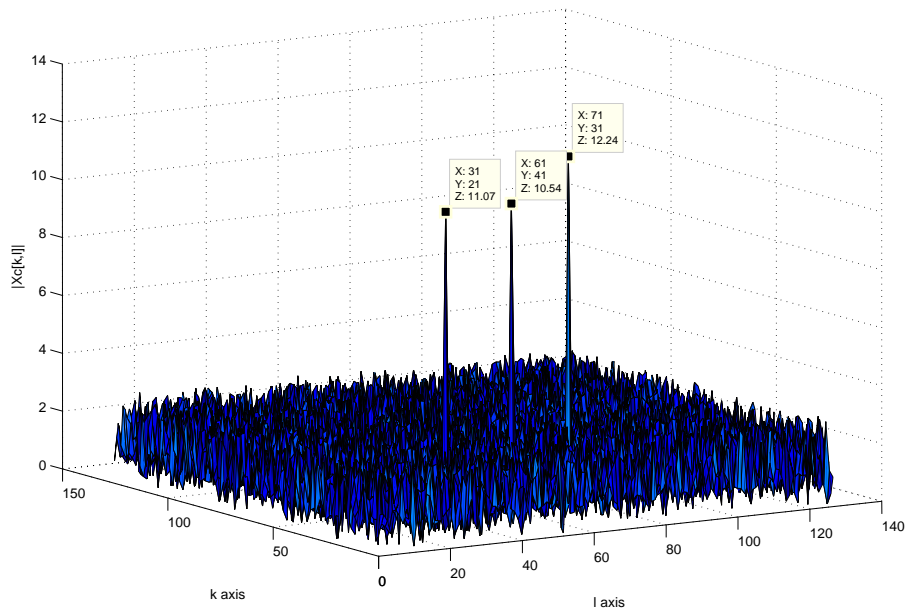


Figure 8. $X_{C_8} - (k_0 = 20, l_0 = 30); (k_1 = 40, l_1 = 60); (k_2 = 30, l_2 = 70)$

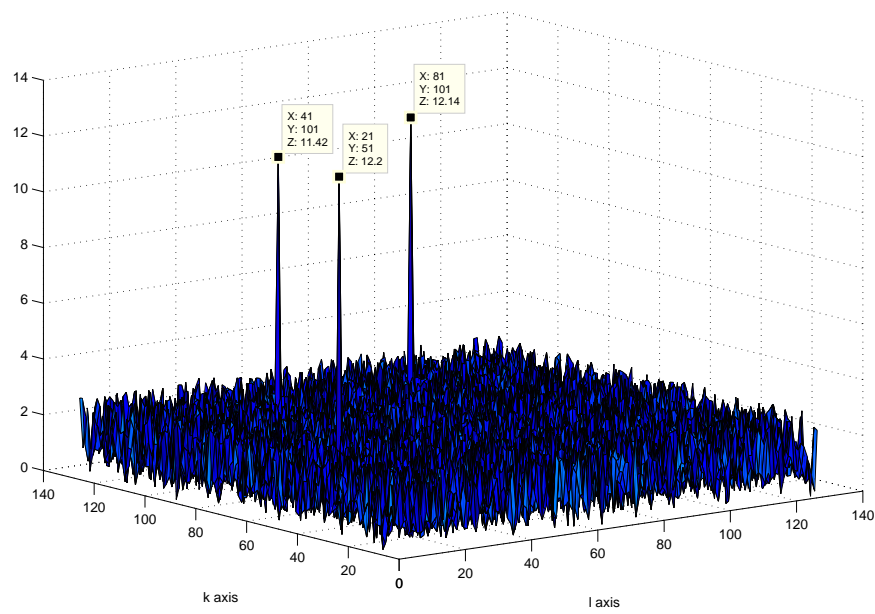


Figure 9. $X_{C_9} - (k_0 = 50, l_0 = 20); (k_1 = 100, l_1 = 40); (k_2 = 100, l_2 = 80)$

4. THE KRONECKER PRODUCTS PROPERTIES

In this section some definitions and properties of Kronecker products are stated.

DEFINITION 4.1. *Let A be an $n \times p$ matrix and B an $m \times q$ matrix.*

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,p-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,p-1} \end{bmatrix} \quad (8)$$

then,

$$A \otimes B = \begin{bmatrix} a_{0,0} \cdot B & a_{0,1} \cdot B & \cdots & a_{0,p-1} \cdot B \\ a_{1,0} \cdot B & a_{1,1} \cdot B & \cdots & a_{1,p-1} \cdot B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} \cdot B & a_{n-1,1} \cdot B & \cdots & a_{n-1,p-1} \cdot B \end{bmatrix}_{m \cdot n \times p \cdot q} \quad (9)$$

The $m \cdot n \times p \cdot q$ matrix is called the Kronecker product of A and B . It is also called the direct product or the tensor product. For square matrices the Kronecker products of the matrices A and B can be formulated as follows:

$$A \otimes B = [a_{k\ell} \cdot B]_{k,\ell \in Z/R} \quad (10)$$

Here the matrix A is assumed to be a square matrix of order R and the matrix B is assumed to be of order S . The order of the product $A \otimes B$ is $R \cdot S = N$. Let A and C be $R \times R$ matrices, and B and D be $S \times S$ matrices. Next, form the tensor products $A \otimes B$ and $C \otimes D$. Then the following result can be obtained through direct matrix multiplication:

$$(A \otimes B)(C \otimes D) = AC \otimes BD \quad (11)$$

If we denote I_R, I_S as identity matrices of order R and S , respectively, we have

$$(A \otimes B) = (A \otimes I_S)(I_R \otimes B) \quad (12)$$

From this expression we can see that the action of computing with the matrix $(A \otimes B)$ can be performed in two stages: An action for the computation of $(I_R \otimes B)$, followed by an action for the computation of $(A \otimes I_S)$.

5. THE DCFT AS A SIGNAL OPERATOR

The set of all the signals of the form:

$$\begin{aligned} x : \mathbb{Z}_N &\rightarrow \mathbb{C} \\ n &\mapsto x[n] \end{aligned} \quad (13)$$

constitutes a signal space denoted by $\ell(\mathbb{Z}_N)$. This signal space is the space of all discrete complex signals of length equal to N . The DCFT can be formulated as an operator that converts a signal of length N into a signal of length N by N .

$$\begin{aligned} \mathcal{G}_h : \ell(\mathbb{Z}_N) &\rightarrow \ell(\mathbb{Z}_N) \times \ell(\mathbb{Z}_N) \\ x &\mapsto \mathbf{X}_N = \mathcal{G}_h \{x\} \end{aligned} \quad (14)$$

where,

$$\mathbf{X}_N[k, l] = (\mathcal{G}_h \{x\}) [k, l] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] W_N^{ln^2 + kn} = X_c[k, l], \quad 0 \leq k, l \leq N-1 \quad (15)$$

From this formulation the DCFT computation can be expressed as a matrix-vector product.

6. THE DCFT MATRIX REPRESENTATION

Taking advantage of the indexing scheme utilized to express the Hadamard product operation between the input signal and the de-chirp factor, the DCFT can be expressed as a matrix-vector product. The formulation is initialized utilizing a discrete signal of length $N = 8$ as an example. Equation 7 can be used as a natural starting point.

$$X_c[k, l] = \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} F(8) \cdot \begin{bmatrix} x[0] \\ x[1]W_N^l \\ x[2]W_N^{4l} \\ x[3]W_N^{9l} \\ x[4]W_N^{16l} \\ x[5]W_N^{25l} \\ x[6]W_N^{36l} \\ x[7]W_N^{49l} \end{bmatrix} \quad (16)$$

$$X_c[k, l] = \mathbf{X}_{ckn} \cdot [x[n] \odot_N W_N^{ln^2}] \quad (17)$$

where \odot denotes the Haddamard product. Replacing k with its index variation converts X_{ckn} in a Fourier Matrix $F(N) = F_N$ of order N .

$$X_{ckn} = F_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 & W_N^4 & W_N^5 & W_N^6 & W_N^7 \\ 1 & W_N^2 & W_N^4 & W_N^6 & 1 & W_N^2 & W_N^4 & W_N^6 \\ 1 & W_N^3 & W_N^6 & W_N^1 & W_N^4 & W_N^7 & W_N^2 & W_N^5 \\ 1 & W_N^4 & 1 & W_N^4 & 1 & W_N^4 & 1 & W_N^4 \\ 1 & W_N^5 & W_N^2 & W_N^7 & W_N^4 & W_N^1 & W_N^6 & W_N^3 \\ 1 & W_N^6 & W_N^4 & W_N^2 & 1 & W_N^6 & W_N^4 & W_N^2 \\ 1 & W_N^7 & W_N^6 & W_N^5 & W_N^4 & W_N^3 & W_N^2 & W_N^1 \end{bmatrix} \quad (18)$$

From Equation 17, it can be seen that, for $l = 0$, the DCFT reduces to the DFT. For $l \neq 0$, the general formulation of the DTFT is obtained for a discrete signal $x \in \ell(\mathbb{Z}_N)$.

7. COMPUTER CLUSTER IMPLEMENTATION OF MODULATED DCFT METHOD

The modulated DCFT method consists of taking the DFT of the Hadamard product of a signal $x[n]$ of length N and the de-chirp factor with phase proportional to ln^2 . The DFT, in turn, is computed with any of the many efficient fast Fourier transform (FFT) algorithms such as the Cooley-Tukey or Gentleman-Sande. This modulated DCFT method was implemented in a high performance instructional computer cluster at the University of Puerto Rico, Mayaguez Campus. This implementation uses the flowchart shown in Figure 10. The implementation utilized the open source FFTW software package *FFTW* - (<http://www.fftw.org>) developed by Mateo Frigo.² Table 1 presents lengths of various discrete signals whose DCFT was computed using the modulated DCFT method and the FFTW. The table also shows absolute times of computations on the computer cluster.

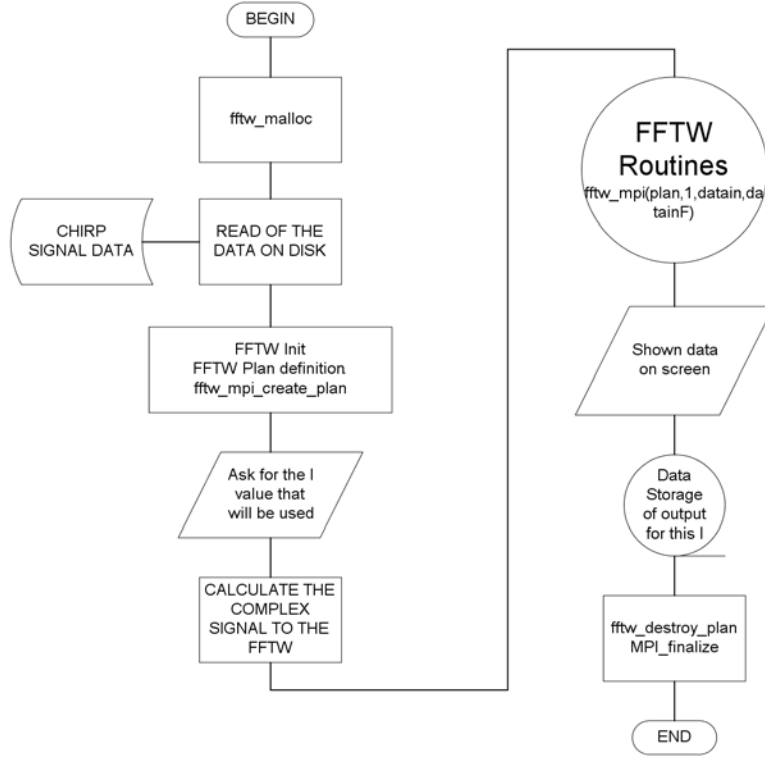


Figure 10. Flow Chart for cluster implementation of the DCFT

Table 1. Benchmark time comparison for Cluster implementation

Samples	Cluster (seconds)
64	0
128	0.05
256	0.21
512	0.97
1024	4.19
2048	17.72
4096	74.06
8192	310.20

8. KRONECKER FACTORS HADAMARD PRODUCT (KFHP) METHODS FOR DCFT ALGORITHM FORMULATION

This section presents two methods for the computation of the DCFT. These methods take advantage of properties of Kronecker products algebra, a branch of finite dimensional multilinear algebra to decompose into a sequence of factors the Hadamard product prevalent in the DCFT formulation. In essence, the methods concentrate on the computation of the Hadamard product of the return signal and the de-chirping or nonlinear demodulation through iterative methods which help reduce pre-computational storage and data transfer, two major factors influencing computational burden for very long discrete signals. The first method concentrates on signals whose length is a power of 2.

The second method treats signals of prime length. This latter method follows the work of Xia³ and Xia¹ who has demonstrated the advantage of using discrete signals of prime length since, for example, the peak value of the mainlobe of the DCFT of a single quadratic chirp is \sqrt{N} times the value of the sidelobes. As presented at the introduction, an algorithm is defined in this work as a composition of linear operators. Matrix representation of linear operators are obtained with respect to the standard basis in order to analyze its computational structure and effect an algorithm formulation. An algorithm, in turn, can be redefined as the decomposition of the matrix representation of an linear operator into a sequence of factor matrices. A computational method is also defined as a structured set of algorithms.

8.1. KFHP Power of Two Method for DCFT Computation

The KFHP power of two method is formulated in this section using Equation 17, again, as the starting point. A great advantage of developing the KFHP power of two method is that for discrete signals whose length is a power of two, there exist very efficient algorithms for the computation of the DFT which follows the Hadamard product in every DCFT computation. The formulation starts as follows:

Let

$$x_{l_N}[n] = x[n] \odot_N W_N^{ln^2} \quad (19)$$

, and let,

$$\psi_{l_N}[n] = W_N^{ln^2} = \exp\left(-\frac{j2\pi ln^2}{N}\right) \quad (20)$$

Taking ln^2 from $\psi_{l_N}[n]$, the following matrix can be constructed:

$$\Psi_{l_N}[n] = [ln^2]_{l,n \in \mathbb{Z}_N} \quad (21)$$

For the specific length $N = 8$ the following result is obtained:

$$\Psi_{l_8}[n] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 4 & 8 & 12 & 16 & 20 & 24 & 28 \\ 0 & 9 & 18 & 27 & 36 & 45 & 54 & 63 \\ 0 & 16 & 32 & 48 & 64 & 80 & 96 & 112 \\ 0 & 25 & 50 & 75 & 100 & 125 & 150 & 175 \\ 0 & 36 & 72 & 108 & 144 & 180 & 216 & 252 \\ 0 & 49 & 98 & 147 & 196 & 245 & 294 & 343 \end{bmatrix} \quad (22)$$

Performing *modulo* – 8 arithmetic results in

$$\Psi_{l_8}[n] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix} \quad (23)$$

Equation 23 is an important result since it presents the quadratic values of the twiddle or phase factors which can be expressed as a product of sparse matrices. From the matrix $\Psi_{l_N}[n]$ the matrix $C_{l_N}^2[n]$ can be obtained. This matrix is the twiddle factor of the quadratic variation of the DCFT. Performing the Hadamard product of each column of $C_{l_N}^2[n]$ with the discrete signal $x[n]$. Finally, the calculation of the DFT of the complex sequences. $C_{l_N}^2[n]$ is given by the following expression for $N = 8$:

$$C_{l_8}^2[n] = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \end{bmatrix} \quad (24)$$

To obtain generalization of the $C_{l_N}^2[n]$ matrix, the basic matrix $C_{l_2}^2[n]$ can be used as a starting point:

$$C_{l_2}^2[n] = \begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix} \quad (25)$$

with

$$C_{l_4}^2[n] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \end{bmatrix} \quad (26)$$

Applying the permutation matrix $P_{4,2}$ results in

$$C_{l_4}^{2'}[n] = C_{l_4}^2[n] \cdot P_{4,2} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^2 & W_4 & W_4^3 \\ 1 & 1 & 1 & 1 \\ 1 & W_4^2 & W_4 & W_4^3 \end{bmatrix} \quad (27)$$

Proceeding to factor

$$C_{l_4}^{2'}[n] = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & W_4^2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & W_4^1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & W_4^2 \end{bmatrix} \end{bmatrix} \quad (28)$$

$$C_{l_4}^{2'}[n] = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & W_4^1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix} \end{bmatrix} \quad (29)$$

Through direct factorization the following Kronecker products decomposition is obtained which resembles a decomposition of the discrete Fourier transform (DFT) matrix:

$$C_{l_4}^{2'}[n] = \left[\begin{bmatrix} I_2 & I_2 \\ I_2 & I_2 \end{bmatrix} \begin{bmatrix} I_2 & \\ & D_{4,2}^c \end{bmatrix} \begin{bmatrix} C_{l_2}^2 & \\ & C_{l_2}^2 \end{bmatrix} \right] \quad (30)$$

Setting $R = 2$ and $S = 2$, the following result is obtained

$$C_{l_4}^{2'}[n] = (\text{ones}[2] \otimes I_2) T_{2,2}^c (I_2 \otimes C_{l_2}^2) \quad (31)$$

where,

$$D_{N,n}^c = \begin{bmatrix} 1 & & & & & \\ & W_N & & & & \\ & & W_N^4 & & & \\ & & & \ddots & & \\ & & & & & W_N^{(n-1)^2} \end{bmatrix} \quad (32)$$

and

$$T_{R,S}^c = \begin{bmatrix} (D_{N,R}^c)^0 & \\ & (D_{N,R}^c)^1 \end{bmatrix} \quad (33)$$

Finally,

$$\text{ones}[2] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (34)$$

As an example, using for $N = 8$, $R = 4$, and $S = 2$ results in

$$C_{l_8}^2[n] = (\text{ones}[2] \otimes I_4) T_{4,2}^c (I_2 \otimes C_{l_4}^2) \quad (35)$$

Through direct computation, a general form of the Equation 35 is formulated as follows:

THEOREM 8.1. *For N is a power of 2,*

$$C_{l_N}^2[n] = (\text{ones}[2] \otimes I_{\frac{N}{2}}) T_{\frac{N}{2},2}^c (I_2 \otimes C_{l_{\frac{N}{2}}}^2) \quad (36)$$

where $T_{\frac{N}{2},2}^c$ is defined in Equation 33

8.2. KFHP Prime Length Method for DCFT Computation

As stated above, DCFT computation for discrete signals of prime length is of great importance due to the fact that the mainlobe of quadratic chirps are better estimated. Below a formulation of the Kronecker factors Hadamard product (KFHP) is developed for discrete signals of prime length.

$$C_3[l] = \left[\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3 & W_3^2 \\ 1 & W_3^2 & W_3 \end{bmatrix}, \begin{bmatrix} 1 & W_3 & W_3 \\ 1 & W_3^2 & 1 \\ 1 & 1 & W_3^2 \end{bmatrix}, \begin{bmatrix} 1 & W_3^2 & W_3^2 \\ 1 & 1 & W_3 \\ 1 & W_3 & 1 \end{bmatrix} \right] \quad (37)$$

Using the integer 2 as a generator for the multiplicative group $G_{3,2}$,⁴ the following expression is obtained.

$$\begin{aligned} G_{3,2} &= \{\langle 2^1 \rangle_3, \langle 2^2 \rangle_3\} \\ G_{3,2} &= \{2 \ 1\} \end{aligned}$$

This expression has the following associated permutation matrix:

$$P_{G_{3,2}} = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right] \quad (38)$$

Applying this permutation to all to rows and columns of the C_3 matrix results in

$$C_{3r}[l] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3 & W_3^2 \\ 1 & W_3^2 & W_3 \end{bmatrix}, \begin{bmatrix} 1 & W_3 & W_3 \\ 1 & W_3^2 & 1 \\ 1 & 1 & W_3^2 \end{bmatrix}, \begin{bmatrix} 1 & W_3^2 & W_3^2 \\ 1 & 1 & W_3 \\ 1 & W_3 & 1 \end{bmatrix} \quad (39)$$

This also produces the following result

$$C_{3r}[l] = \left[C_{3r}[0] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C_{3r}[0] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & W_3^{-1} & 0 \\ 0 & 0 & W_3^{-1} \end{bmatrix}, C_{3r}[0] \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & W_3^{-2} & 0 \\ 0 & 0 & W_3^{-2} \end{bmatrix} \right] \quad (40)$$

$$C_{3r}[l] = [C_{3r}[0] \cdot H_3[0], C_{3r}[0] \cdot H_3[1], C_{3r}[0] \cdot H_3[2]] \quad (41)$$

where,

$$H_3[l] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & W_3^{-l} & 0 \\ 0 & 0 & W_3^{-l} \end{bmatrix} \quad (42)$$

It is worth noticing that the structure of the $C_{3r}[0]$ matrix resembles a Hankel matrix.

$$C_{3r}[0] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3 & W_3^2 \\ 1 & W_3^2 & W_3 \end{bmatrix} \quad (43)$$

Removing the $0th$ row and $0th$ column, the following result is obtained:

$$C'_3[0] = \begin{bmatrix} W_3 & W_3^2 \\ W_3^2 & W_3 \end{bmatrix} \quad (44)$$

The process can be repeated for $N = 5$. The integer 3 can be used as a generator for $G_{5,3}$.

$$\begin{aligned} G_{5,3} &= \{ \langle 3^1 \rangle_5, \langle 3^2 \rangle_5, \langle 3^3 \rangle_5, \langle 3^4 \rangle_5 \} \\ G_{5,3} &= \{ 3 \ 4 \ 2 \ 1 \} \end{aligned}$$

The following permutation matrix is then associated with this group:

$$P_{G_{5,3}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (45)$$

Then, $C_5[l]$ is defined as follows:

$$C_5[l] = \left[\begin{array}{c} \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & W_5 & W_5^2 & W_5^3 & W_5^4 \\ 1 & W_5^2 & W_5^4 & W_5 & W_5^3 \\ 1 & W_5^3 & W_5 & W_5^4 & W_5^2 \\ 1 & W_5^4 & W_5^3 & W_5^2 & W_5 \end{array} \right] \\ \left[\begin{array}{ccccc} 1 & W_5 & W_5^4 & W_5^4 & W_5 \\ 1 & W_5^2 & W_5 & W_5^2 & 1 \\ 1 & W_5^3 & W_5^3 & 1 & W_5^4 \\ 1 & W_5^4 & 1 & W_5^3 & W_5^3 \\ 1 & 1 & W_5^2 & W_5 & W_5^2 \end{array} \right] \\ \left[\begin{array}{ccccc} 1 & W_5^3 & W_5^2 & W_5^2 & W_5^3 \\ 1 & W_5^4 & W_5^4 & 1 & W_5^2 \\ 1 & 1 & W_5 & W_5^3 & W_5 \\ 1 & W_5 & W_5^3 & W_5 & 1 \\ 1 & W_5^2 & 1 & W_5^4 & W_5^4 \end{array} \right] \\ \left[\begin{array}{ccccc} 1 & W_5^2 & W_5^3 & W_5^3 & W_5^2 \\ 1 & W_5^3 & 1 & W_5 & W_5 \\ 1 & W_5^4 & W_5^2 & W_5^4 & 1 \\ 1 & 1 & W_5^4 & W_5^2 & W_5^4 \\ 1 & W_5 & W_5 & 1 & W_5^3 \end{array} \right] \\ \left[\begin{array}{ccccc} 1 & W_5^4 & W_5 & W_5 & W_5^4 \\ 1 & 1 & W_5^3 & W_5^4 & W_5^3 \\ 1 & W_5 & 1 & W_5^2 & W_5^2 \\ 1 & W_5^2 & W_5^2 & 1 & W_5 \\ 1 & W_5^3 & W_5^4 & W_5^3 & 1 \end{array} \right] \end{array} \right] \quad (46)$$

The $P_{G_{5,3}} \cdot C_5[l] \cdot P_{G_{5,3}}^{-1}$ expression is computed as follows:

$$C_{5r}[l] = \left[\begin{array}{c} \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & W_5 & W_5^2 & W_5^4 & W_5^3 \\ 1 & W_5^2 & W_5^4 & W_5^3 & W_5 \\ 1 & W_5^4 & W_5^3 & W_5 & W_5^2 \\ 1 & W_5^3 & W_5 & W_5^2 & W_5^4 \end{array} \right] \\ \left[\begin{array}{ccccc} 1 & W_5 & W_5^4 & W_5 & W_5^4 \\ 1 & W_5^2 & W_5 & 1 & W_5^2 \\ 1 & W_5^3 & W_5^3 & W_5^4 & 1 \\ 1 & 1 & W_5^2 & W_5^2 & W_5 \\ 1 & W_5^4 & 1 & W_5^3 & W_5^3 \end{array} \right] \\ \left[\begin{array}{ccccc} 1 & W_5^3 & W_5^2 & W_5^3 & W_5^2 \\ 1 & W_5^4 & W_5^4 & W_5^2 & 1 \\ 1 & 1 & W_5 & W_5 & W_5^3 \\ 1 & W_5^2 & 1 & W_5^4 & W_5^4 \\ 1 & W_5 & W_5^3 & 1 & W_5 \end{array} \right] \\ \left[\begin{array}{ccccc} 1 & W_5^2 & W_5^3 & W_5^3 & W_5^2 \\ 1 & W_5^3 & 1 & W_5 & W_5 \\ 1 & W_5^4 & W_5^2 & 1 & W_5^4 \\ 1 & 1 & W_5^4 & W_5^2 & W_5^4 \\ 1 & W_5 & W_5 & 1 & W_5^3 \end{array} \right] \\ \left[\begin{array}{ccccc} 1 & W_5^4 & W_5 & W_5^4 & W_5 \\ 1 & 1 & W_5^3 & W_5^3 & W_5^4 \\ 1 & W_5 & 1 & W_5^2 & W_5^2 \\ 1 & W_5^3 & W_5^4 & 1 & W_5^3 \\ 1 & W_5^2 & W_5^2 & W_5 & 1 \end{array} \right] \end{array} \right] \quad (47)$$

The transformation matrix H_5 for the $C_{5r}[l]$ expression results in,

$$H_5[l] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & W_5^{-l} & 0 & 0 & 0 \\ 0 & 0 & W_5^{-4l} & 0 & 0 \\ 0 & 0 & 0 & W_5^{-l} & 0 \\ 0 & 0 & 0 & 0 & W_5^{-4l} \end{bmatrix} \quad (48)$$

where the inner matrix is:

$$C'_5[l] = \begin{bmatrix} W_5 & W_5^2 & W_5^4 & W_5^3 \\ W_5^2 & W_5^4 & W_5^3 & W_5 \\ W_5^4 & W_5^3 & W_5 & W_5^2 \\ W_5^3 & W_5 & W_5^2 & W_5^4 \end{bmatrix} \quad (49)$$

This result relates to several methods for computing Hankel matrix-vector products presented by Gumerov.⁵

Thus,

$$C_{5r}[l] = P_{G_{5,3}} \cdot C_5[l] \cdot P_{G_{5,3}}^{-1} \quad (50)$$

$$C_5[l] = P_{G_{5,3}}^{-1} \cdot C_{5r}[l] \cdot P_{G_{5,3}} \quad (51)$$

Now, the matrix-vector product for the signal data results in

$$C_5[l] \cdot x[n] = P_{G_{5,3}}^{-1} \cdot C_{5r}[l] \cdot P_{G_{5,3}} \cdot x[n] \quad (52)$$

As a general formulation, for N any prime number, the following theorem is stated

THEOREM 8.2. *If N is a prime number and k an integer generator of $G_{N,k} = \{\langle k^1 \rangle_N, \langle k^2 \rangle_N, \langle k^3 \rangle_N, \dots, \langle k^{N-1} \rangle_N\}$, then*

$$C_N[l] = P_{G_{N,k}}^{-1} \cdot C_{Nr}[1] \cdot H_N[l] \cdot P_{G_{N,k}} \quad (53)$$

Proof. The proof of the theorem is presented by direct computation, in the following manner:

The permutation operation is defined as:

$$\sigma_{G_{N,k}} = \begin{bmatrix} 0 & 1 & 2 & 3 & \dots & N-1 \\ 0 & \langle k^1 \rangle_N & \langle k^2 \rangle_N & \langle k^3 \rangle_N & \dots & \langle k^{N-1} \rangle_N \end{bmatrix}$$

And the transformation matrix $H_N[l]$ is:

$$H_N[l] = P_{G_{N,k}} \cdot \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & W_N^{-l} & \dots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & W_N^{-(N-1)l} \end{bmatrix} \cdot P_{G_{N,k}}^{-1} \quad (54)$$

□

9. CONCLUSION

This work centered on the use of computational signal algebra techniques for the design and development of discrete chirp Fourier transform (DCFT) algorithms for active remote sensing applications such as synthetic aperture radar signal processing. The DCFT algorithms were formulated using Kronecker products algebra, a branch of finite dimensional multilinear algebra. Group theoretic properties of input and output indexing sets were utilized for the decomposition of matrix representations of DCFT formulations. algorithm implementations were conducted on high performance clusters for very long discrete complex signals modeling multicomponent chirp signal returns.

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